Patterns and statistics for shifts of finite type

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Setup: given a binary word x, say x *avoids* a pattern $w \in \{0,1\}^k$ if x does not contain w as a subword, i.e. if for all i,

$$x_i x_{i+1} \cdots x_{i+k} \neq w_1 w_2 \cdots w_k$$

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Warmup

Let B_n be the set of length n words that avoid 1001.

Exponential growth rate of $|B_n|$?

Density of 1s in a typical element of B_n? ($>, <$ or $= \frac{1}{2}$?)

Follower set graph construction, enumeration of B_n

Sequences avoiding 1001 $\leftrightarrow$ paths in G_{1001}

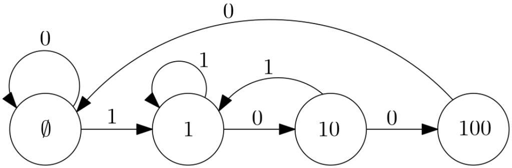
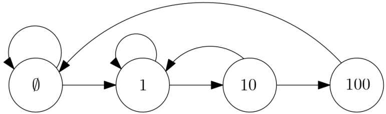


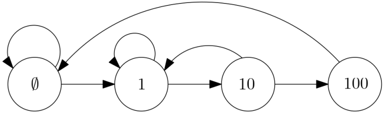
Figure: Vertices of G_{1001} are proper prefixes of 1001

Bijection: read the edge labels



$B_n =$ paths in G of length n

$\lim_{n \to \infty} \frac{1}{n} \log |B_n| =$ Perron-Frobenius eigenvalue of $G \approx 1.867$



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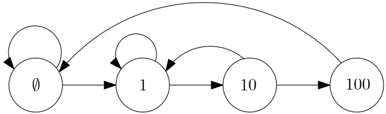
Uniform measure on paths induces a Markov chain on G:

$$P_{ij} = A_{ij} \frac{r_j}{\lambda r_i}$$

$A =$ adj matrix of G

$\lambda, r =$ PF eigenvalue/eigenvector



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Compute stationary measure μ for P

asymptotic density of 1s $= \mu(1) =$ 👻

General setup: alphabet $[q]$ on $\mathbb{Z}^d$

Pattern avoiding

A **pattern** is any map $w : K \to [q]$ for a finite $K \subset \mathbb{Z}^d$. A configuration $x : \mathbb{Z}^d \to [q]$ avoids w if it does not contain any translation of w.

Shift of finite type

Fix a finite family $\mathcal{F}$ of patterns on $\mathbb{Z}^d$. The **shift of finite type** $X = X(\mathcal{F})$ is the set of all configurations $x : \mathbb{Z}^d \to [q]$ that avoid all patterns in $\mathcal{F}$.

Let $X = X(\mathcal{F})$ be a shift of finite type

For a box V, $X^V =$ configurations on V that can occur in elements of X

Entropy

The entropy exists:

$$h(X) := \lim_{V \uparrow \mathbb{Z}^d} \frac{1}{|V|} \log |X^V| \in [0, \log q).$$

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Theorem (Measure of maximal entropy)

There is a probability measure on X which attains the maximum possible entropy $h(X)$ (both measure-theoretic and topological).

On $\mathbb{Z}$, it's always unique and Markovian (Perron-Frobenius construction)

Large deviations/Gibbs measure formulation

Fix patterns $\mathcal{F}$, weights $\beta \in \mathbb{R}^{\mathcal{F}}$.
For $x : V \to [q]$, set

$$\mu^{V}_{\mathcal{F},\beta}(x) \sim \exp\left(\sum_{w \in \mathcal{F}} -\beta_w N_w(x)\right),$$

$N_w(x) =$ number of copies of w in x.

Thermodynamic limit

As $V \uparrow \mathbb{Z}^d$, $\mu^{V}_{\mathcal{F},\beta}$ converges to a probability measure $\mu_{\mathcal{F},\beta}$ on $[q]$-configurations on $\mathbb{Z}^d$.

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On $\mathbb{Z}$, can describe μ by a transfer matrix

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As $V \uparrow \mathbb{Z}^d$, $\mu^V_{\mathcal{F},\beta}$ converges weakly to a measure $\mu_{\mathcal{F},\beta}$ on $[q]$-configuraitons on $\mathbb{Z}^d$.

$\beta \to \infty$: measure of maximal entropy for the shift space $X(\mathcal{F})$
$\beta = 0$: iid Uniform on $[q]$
$\beta \to -\infty$: packing with tiles $\mathcal{F}$

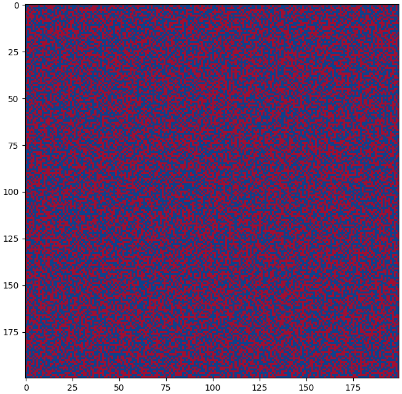

Figure: Sample from $\mu_{\mathcal{F},\beta}$ with $\mathcal{F} = \left\{ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}, \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right\}$, and $\beta = (1, 1)$.

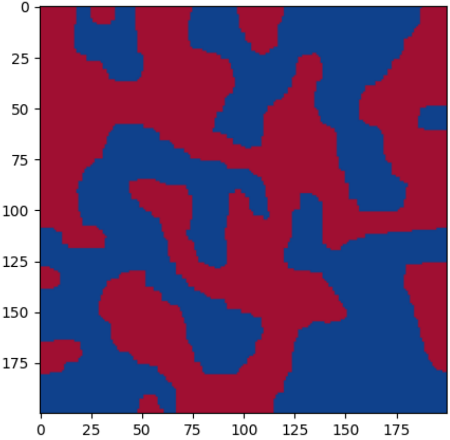

Figure: Sample from $\mu_{\mathcal{F},\beta}$ with $\mathcal{F} = \left\{ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}, \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right\}$, and $\beta = (-4, -4)$.

Spooky pattern set $\mathcal{F} = \{$🎃, 🦇, 💀, 🧙$\}$, $\beta \to -\infty$?

Figure: Sample from $\mu_{\mathcal{F},\beta}$ with $\mathcal{F} = \{$🎃, 🦇, 💀, 🧙$\}$, $\beta \to -\infty$

Question: How is the shift space X or the Gibbs measure μ controlled by combinatorial structure of $\mathcal{F}$?

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- Ordering shift spaces by entropy
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- Pattern densities; correlations

Entropy and hitting time on $\mathbb{Z}$: Abracadabra!

Fix a word $w \in [q]^k$ ($\mathcal{F} = \{w\}$)

$\tau_w =$ hitting time of w:

$$\tau_w(x) = \min\{t > 0 : x_{t-k+1}x_{t-k+2}\cdots x_t = w_1 w_2 \cdots w_k\}.$$

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Auto-correlation polynomial:

$$\phi_w(z) = \sum_{j \in \mathcal{O}(w,w)} z^j,$$

where $\mathcal{O}(w, w) = \{j : w_1 w_2 \cdots w_j = w_{k-j+1} w_{k-j+2} \cdots w_k\}$.

e.g. $\phi_{111}(z) = z^3 + z^2 + z$; $\phi_{1001}(z) = z^4 + z$.

Abracadabra martingale

Under iid Uniform($[q]$) measure,

$$\mathbb{E}\tau_w = \phi_w(q).$$

The same statistic controls the entropy:

Theorem (Guibas-Odlyzko '81)

For patterns w, w' on $\mathbb{Z}$, TFAE:

1. $\mathbb{E}\tau_w \leq \mathbb{E}\tau_{w'}$
2. $\tau_w \prec_{stoc} \tau_{w'}$
3. $h(X(w)) \leq h(X(w))$

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Proof: compute with recursions

Nicer proof under a condition on follower set graphs $G_w, G_{w'}$

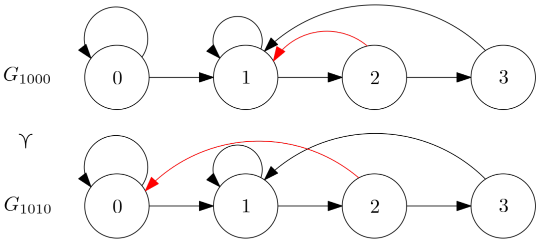

Figure: $\phi_{1000}(z) = z^4, \phi_{1010}(z) = z^4 + z^2$, so $\tau_{1000} \prec_{\text{stoc}} \tau_{1010}$.

$G_w \succ G_{w'}$ if for every j, you can pair the outgoing edges at j in G_w with those at j in $G_{w'}$ so that the edges in G_w go further to the right.

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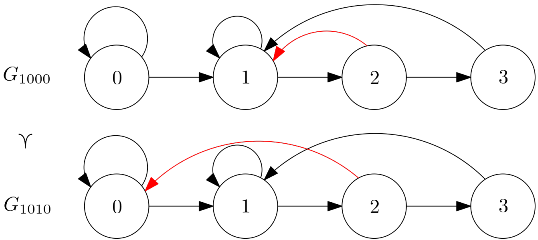

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Theorem (Chandgotia-Marcus-R.-Wu '24)

$G_w \succ G_{w'} \implies h(X(w)) < h(X(w'))$ *and* $\tau_w \prec_{stoc} \tau_{w'}$.

Proof of $G_w \succ G_{w'} \implies \tau_w \prec_{\text{stoc}} \tau_{w'}$

Couple simple random walks ξ, ξ' on $G_w, G_{w'}$:

- If $\xi = \xi'$, they move together
- If $\xi' < \xi$, then ξ freezes while ξ' moves independently

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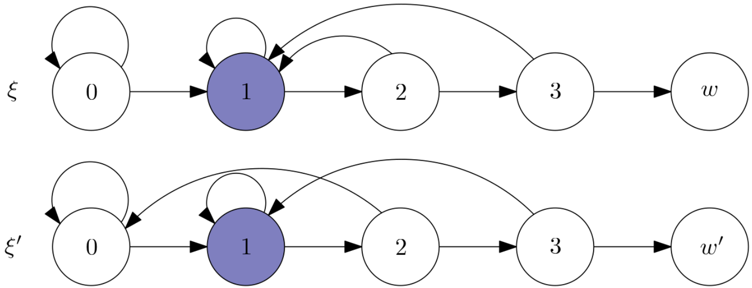

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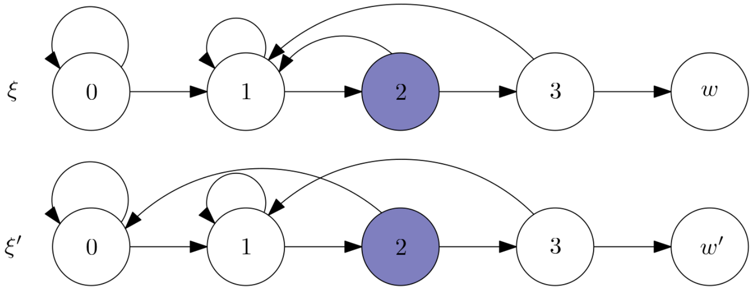

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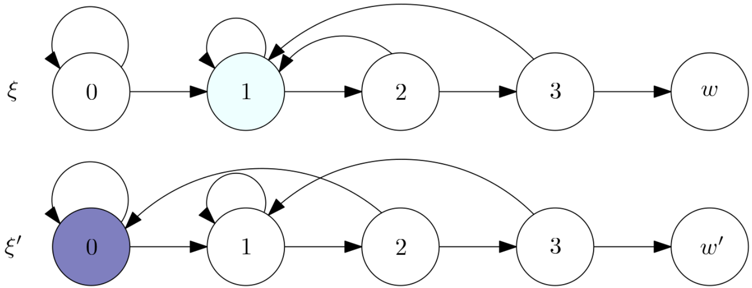

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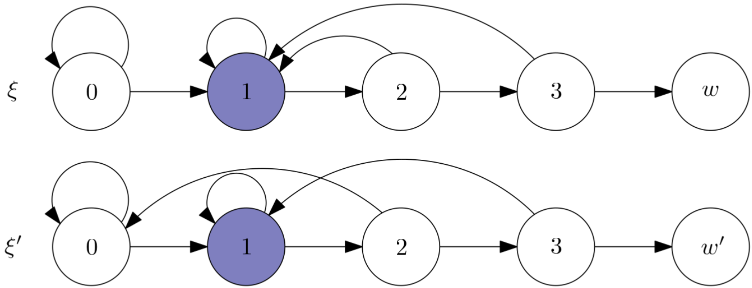

Proof of $G_w \succ G_{w'} \implies h(X(w)) < h(X(w'))$

Motonicity of the right Perron eigenvector:

Lemma (CMRW '24)

The entries of the right eigenvector r of G_w strictly decrease exponentially:

$$\frac{r_{j+1}}{r_j} \leq h(X(w)) - q + 1 \in (0, 1)$$

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Combinatorics of G_w: how does $\prec$ relate to ϕ_w?

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Inclusion-exclusion argument

Not clear how to improve to injections, or conjugacy

We can handle some special cases in 2D, e.g:

$$w = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad w' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

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Lexicographic replacement surjection $X(w') \to X(w)$

$+$ **entropy minimality** $\implies h(X(w)) < h(X(w'))$

Extender set of a finite word $w \in X$ is all pairs (a, b) of one-sided infinite words such that $awb \in X$.

Conjecture

Let $X = X(\mathcal{F})$ be any 1D shift of finite type, w, w' allowable words in X. If w, w' have the same extender set, then TFAE:

- $h(X(\mathcal{F} \cup \{w\})) \leq h(X(\mathcal{F} \cup \{w'\}))$
- $\phi_w(h(X)) \leq \phi_{w'}(h(X))$
- $\mathbb{E}_X(\tau_w) \leq \mathbb{E}_X(\tau_{w'})$

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Heuristic: $\tau_w \approx$ Exponential, escape rate $=$ entropy loss

Moment generating function of τ_w involves ϕ_w

Conjugacy problem

Isomorphism of shifts

Let X, Y be 1D shift spaces. A **conjugacy** $f : X \to Y$ is any bijection of the form

$$f(x)_i = F(x_{i-m}, x_{i-m+1}, \cdots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+a})$$

for some function F, and $m, a \geq 0$.

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for some function F, and $m, a \geq 0$.

Conjugacies are local. 'Sliding block code,' $m =$ memory, $a =$ anticipation

In general, computing conjugacy classes is difficult!

Theorem (CMRW '24)

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Open:

- for 1D shifts $X(w)$, is $\phi_w = \phi_{w'}$ equivalent to conjugacy?
- Are $X(110110)$ and $X(100100)$ conjugate? (Note: reversal is not a conjugacy!)

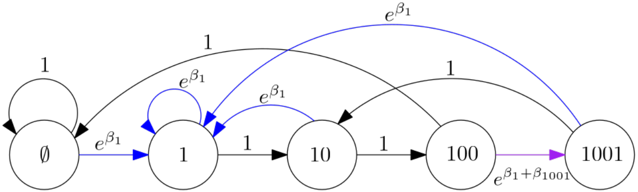
Gibbs measure $\mu_{\mathcal{F},\beta}$

Limit free energy $p_{\mathcal{F}}(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z^n(\beta)$,

$$Z^n(\beta) = \sum_{x\in\{0,1\}^n} \sum_{w\in\mathcal{F}} \exp\left(-\beta_w N_w(x)\right)$$

Derivatives of p $\to$ expectations of observables wrt μ

Transfer matrix from weighted graph, e.g $\mathcal{F} = \{1, 1001\}$:



$$A_{\mathcal{F}}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ e^{\beta_1} & e^{\beta_1} & e^{\beta_1} & 0 & e^{\beta_1} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\beta_1} e^{\beta_{1001}} & 0 \end{pmatrix}$$

$p_{\mathcal{F}}(\beta)$ is the (log of the) Perron-Frobenius eigenvalue of $A_{\mathcal{F}}(\beta)$

Lemma

For any $g : \{0,1\}^n \to \mathbb{R}$ *and* $w \in \mathcal{F}$,

$$\frac{\partial}{\partial \beta_w} \mathbb{E}_\beta[g] = Cov_\beta(g, N_w).$$

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Can compute everything at $\beta = 0$, but it's not always helpful:

$$\mathrm{Cov}_{\beta=0}(N_1, N_{1001}) = O(1).$$

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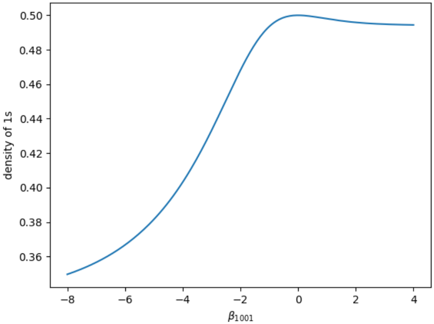
Limit theory for N_w: *Analytic Pattern Matching, Jacquet-Szpankowski*

Joint CLT for $(N_w)_{w \in \mathcal{F}}$?

Note $\frac{\partial}{\partial \beta_v} p(\beta) = \mathbb{P}_\beta(1\{x_0 \cdots = v\}) =$ density of v's

Hope

$\frac{\partial}{\partial \beta_w} \mathbb{P}_\beta(1\{x_0 \cdots = v\})$ a monotonic (or constant) function of $\beta_w \in (0, \infty)$.



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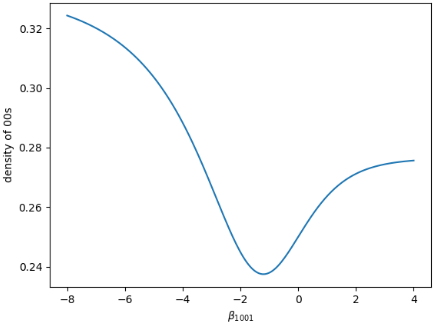

IID Bernoulli(1/2), hitting time τ_w, $N_1(t)$ = number of 1s up to time t

γ_w = asymptotic density of 1s, γ_w^n = density of 1s over $x \in X^n(w)$

Theorem (Maga-R '24+)

If $\gamma_w^n < \frac{1}{2}$ for all n, and $\gamma_w < \frac{1}{2}$, then

$$\mathbb{E}\left[\frac{N_1(\tau_w)}{\tau_w}\right] \geq \frac{1}{2}.$$

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γ_w lives in the shift, $N_1(\tau_w)$ lives in the complement of the shift

Abracadabra martingale is robust, but has limits

Wishlist:

- Simple combinatorial formula for ordering by density of 1s
- Hitting time beyond $\mathbb{Z}$
- Entropy in non-amenable setting

References

- *Shifts of Finite Type Obtained by Forbidding a Single Pattern*. Nishant Chandgotia, Brian Marcus, JR, Chengyu Wu. arXiv:2409.09024
- *Word length, bias and bijections in Penney's ante*. Mathew Drexel, Xuanshan Peng, JR. arXiv:2409.19195
- *String overlaps, pattern matching, and nontransitive games.* L.J Guibas, A.M Odlyzko. Journal of Combinatorial Theory, Series A, 1981.
- *A Martingale Approach to the Study of Occurrence of Sequence Patterns in Repeated Experiments*. Shuo-Yen Robert Li. Annals of Probability, 1980.