

Collisions of random walks and related diffusions

Jacob Richey *

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Abstract

Consider n brownian motion particles on \mathbb{R} , \mathbb{S}^1 , or n discrete time random walks on a graph G . A ‘collision’ occurs when two particles have the same position at the same time. Given a rule for modeling interactions between particles, we can analyze the long term behavior of such a system. We focus on the case where particles are independent, and briefly discuss variants, such as the situation where particles are conditioned not to collide; or when there is a repelling drift term between particles. We derive new formulas for collision times among two or three simple random walks on \mathbb{Z} . Also, we review the existing literature to gain insight about other underlying graphs, and the case of multiple brownian particles.

1 Introduction

Collisions among random walks have been studied in a variety of contexts. In the discrete case, Sousi, Peres and Barlow [1] gave the first steps to understanding the connection between the underlying graph G and questions about collisions among multiple random walks on G : specifically, they investigate what conditions are necessary to guarantee two random walks collide infinitely often almost surely. We give a short account of their results in section 3. The other question that has gained interest in the discrete case is about hitting times and cover times: given a graph G and a vertex subset $A \subset G$, by how much is the hitting time of A or the cover time of A by random walkers decreased as the number of particles increases [5] [4]?

In the discrete case, all kinds of rare events, such as triple collisions, will occur eventually with positive probability. In the continuous regime with brownian particles, the first important question is: do triple collisions occur? Tomoyuki and Ioannis [10] give a criterion in terms of the drift and covariance structure to rule out triple collisions. In a slightly more general setting, where particles may have drifts that depend on their rank relative to other particles, Andrey Sarantsev [15] gives an elegant condition in terms of the particle drifts and variances to rule out multiple collisions, which we summarize in section 4.

Another natural question of interest is: what does the system look like if we condition on the particles *never* colliding [14]? In particular, what is the asymptotic probability of having no collision for a long time[8]? For brownian motions on the circle, Hobson and Werner [9] give an explicit description of the transition kernel for the process conditioned to have no collisions. David Grabner [8] shows that the probability that no collision happens up to time t is a constant multiple of $t^{-n(n-1)/4}$, and explicitly computes the distribution of the particles conditioned on this event as a Bessel process.

*University of Washington; jfrichey@uw.edu.

The statistical physics literature also has some work on annihilating particle systems: for example, the authors in [6] carried out explicit monte carlo simulations of such a system, and were able to recover predicted equilibrium states for the temperature of the system over long time scales, using asymptotics involving collisions between particles. Cepa and Lepingle [2] show that, when particles have a space-dependent repulsion drift term – corresponding to, say, positively charged quantum particles on a line – no triple collisions occur.

Collisions of multiple particles in one dimension is equivalent to the exit time of a single high dimensional particle from a set, so analyzing collisions boils down to understanding such exit times. This is essentially the strategy of all the papers considering Brownian particles mentioned above. Ciesielski and Taylor’s 1962 paper [3] is a classic reference for such problems. For example, the authors explicitly compute exit times from spheres, and the total time spent in a sphere before escaping to infinity. Computing exit times in general is a difficult question, and is essentially equivalent to solving a stochastic boundary problem on a high dimensional region.

2 Simple random walk

We start with some simple calculations regarding simple random walks in \mathbb{Z} , which were known previously [1]. Consider two independent simple random walks in discrete time X^1, X^2 starting from 0. The first natural question to ask about particle collisions is: what rate do collisions occur at? In other words, if $C_t = |\{s \in [1, t] : X_s^1 = X_s^2\}|$ is the number of collisions up to time t , then what is the distribution of C_t ? The expected value $\mathbb{E}C_t$ can be calculated easily by summing over all times up to t :

$$\begin{aligned} \mathbb{E}C_t &= \sum_{s=1}^t \mathbb{P}(X_s^1 = X_s^2) \\ &= \sum_{s=1}^t \sum_{j=-s}^s \mathbb{P}(X_s^1 = X_s^2 = j) \\ &= \sum_{s=1}^t \sum_{j=-s}^s \mathbb{P}(X_s^1 = j)^2. \end{aligned}$$

Viewing the random walk as a path starting from $(0, 0)$ with discrete steps $(1, \pm 1)$, the probability that X^1 is at position j at time s is

$$\mathbb{P}(X_s^1 = j) = 2^{-s} \times |\{\text{paths } (0, 0) \rightarrow (s, j) \text{ with steps } (1, \pm 1)\}|$$

This is only non-zero when j has the same parity as s , in which case the number of such paths is $\binom{s}{s/2+j/2}$. Plugging this in yields

$$\mathbb{E}C_t = \sum_{s=1}^t 4^{-s} \sum_{k=0}^s \binom{s}{k}^2 = \sum_{s=1}^t 4^{-s} \binom{2s}{s} \sim \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

This simple calculation generalizes to the case of n independent simple random walk particles $\{X_t^i : 1 \leq i \leq n\}$, all started from 0:

Proposition 2.1. Let $C_{n,t}$ denote the n -fold collisions of the particles up to time t , i.e. the random variable

$$C_{n,t} = |\{s \in [1, t] : X_s^1 = X_s^2 = \dots = X_s^n\}|.$$

Then for some constants c_n ,

$$\mathbb{E}C_{n,t} \sim c_n \begin{cases} \sqrt{t}, n = 2 \\ \log t, n = 3 \\ 1, n > 3 \end{cases}$$

It makes sense that $\mathbb{E}C_{n,t}$ is always a concave function of t for t large, since particles are less likely to all be co-located for large t values.

Proof sketch: Just as in the case $n = 2$, we have

$$\mathbb{E}C_{n,t} = \sum_{s=1}^t \sum_{j=-s}^s \mathbb{P}(X_s^1 = j)^n = \sum_{s=1}^t 2^{-ns} \sum_{k=0}^s \binom{s}{k}^n.$$

Now approximate asymptotically. □

One immediate consequence of the above formula is:

Corollary 2.2. *Two or three independent simple random walks on \mathbb{Z} are all co-located infinitely often, but four or more independent random walks are only co-located a finite number of times almost surely.*

Proposition 2.1 also gives the asymptotic rate of collisions among any subsets of particles: for example, the expected number of k -subsets of particles that are co-located up to time t is, by linearity of expectation,

$$\mathbb{E} \left| \{(s, S) : s \in [1, t], S \in \binom{[n]}{k}, \text{ and } X_s^i = j \text{ for all } i \in S \text{ and some } j \in [-s, s]\} \right| = \binom{n}{k} \mathbb{E}C_{k,t}.$$

It turns out that for two random walks, we can describe the distribution of $C_t = C_{2,t}$ very precisely, via the following lemma:

Lemma 2.3. *Let X_t^1 and X_t^2 be as above, and set $\tau = \inf\{t > 0 : X_t^1 = X_t^2\}$. Then as $T \rightarrow \infty$,*

$$\mathbb{P}(\tau = T) \sim \frac{2}{\sqrt{\pi}} T^{-3/2}.$$

The proof will actually give us an explicit, though complicated, combinatorial formula for the distribution of τ . The actual distribution can be written down explicitly: we show how this can be done later, after proving proposition 2.4.

Proof. The idea is to look at the distance between the two particles $Z_t = X_t^1 - X_t^2$: Z is a sleepy random walk with steps $-2, 0$, or 2 , with probability $\frac{1}{4}$ of moving to the left or right and probability $\frac{1}{2}$ of staying put, and τ is the first return time of Z to the origin. The return time τ behaves like the first return time of a non-sleepy random walk, but the sleepiness delays it a little bit. Note that after each time where Z takes a ± 2 step, it stays put for a geometric mean 2 amount of time, since $\mathbb{P}(Z_t = Z_{t+1}) = 1/2$.

Let Y_i be iid variables for $i \in \mathbb{N}$ with distribution $P(Y = k) = 2^{-1-k}$ for $k \geq 0$, let B be a Bernoulli $1/2$ random variable, and let $\sigma = \inf\{t > 0 : W_t = 0\}$ be the first return time to 0 for a simple random walk started at $W_0 = 1$. Then we have the distributional equality

$$\tau \stackrel{d}{=} B + (1 - B) \left(1 + \sigma + \sum_{i=1}^{\sigma-1} Y_i \right). \quad (2.1)$$

To explain this, first note that $\tau = 1$ on the event that the first step Z takes is a sleepy step. Otherwise, it moves to ± 2 , in which case we wait for Z to return to the origin. This is the same as waiting for W to return to the origin, plus some number of sleepy steps at each vertex before we return: there are Y_i such steps at each site.

Recall that σ has distribution

$$\mathbb{P}(\sigma = 2s - 1) = 4^{-s} \frac{2}{s+1} \binom{2s}{s},$$

and since $\sum_{i=1}^r Y_i$ is negative binomially distributed,

$$\mathbb{P}\left(\sum_{i=1}^r Y_i = k\right) = 2^{-k-r} \binom{k+r-1}{k}.$$

Now since σ is independent of the Y_i 's, and σ only takes odd values, conditioning on the value of σ yields

$$\begin{aligned} \mathbb{P}\left(\sigma + \sum_{i=1}^{\sigma-1} Y_i = t\right) &= \sum_{s=1}^{\lceil t/2 \rceil} \mathbb{P}(\sigma = 2s - 1) \mathbb{P}\left(\sum_{i=1}^{2s-1} Y_i = t - 2s + 1\right) \\ &= \sum_{s=1}^{\lceil t/2 \rceil} \left(4^{-s} \frac{2}{s+1} \binom{2s}{s}\right) \left(2^{-t} \binom{t-1}{t-2s+1}\right). \end{aligned}$$

Plugging this into (2.1) and approximating asymptotically yields the desired formula. \square

Collisions are a renewal process, and the double collision counter C_t is related to τ in the usual way: if τ_i are iid variables distributed like τ for $i \in \mathbb{N}$, and $\Sigma_i = \sum_{i=1}^j \tau_i$, then

$$C_t = \sup\{j > 0 : \Sigma_j < t\}, \text{ or } C_t = j \text{ when } \Sigma_j < t < \Sigma_{j+1}.$$

That is, we run the particle Z until it returns to 0: this takes time τ , and we have just had a collision. Now the system resets, and we wait again for Z to return, and so on. So the number of collisions seen so far is the number of τ variables up to time t that we have seen.

It is natural to ask about collisions of random walks on arbitrary graphs G , or even arbitrary discrete time Markov chains. As a simple example, we resolve the case of a pair of particles on the

$2m$ -cycle, $m \geq 1$. (We assume the cycle has even length just to avoid particles jumping over each other without colliding.) Let X_t^1, X_t^2 be simple random walks on $\mathbb{Z}/2m\mathbb{Z}$, with $X_0^1 = X_0^2 = 0$: what is the distribution of the first collision time $\tau_{2m} = \inf\{t > 0 : X_t^1 = X_t^2\}$? While it is a bit messy to determine the actual distribution, we can prove:

Proposition 2.4. *The probability generating function of τ_{2m} is given by*

$$\mathbb{E}[w^{\tau_{2m}}] = \frac{w}{2} \left[1 + \frac{m \left(\frac{1}{\sqrt{w}} - \sqrt{\frac{1}{w} - 1} \right)^2}{m - 1 + \left(\frac{1}{\sqrt{w}} - \sqrt{\frac{1}{w} - 1} \right)^{2m}} \right]. \quad (2.2)$$

One can use the fact that the collision hitting times form a renewal process to obtain similar formulas in the case $X_0^1 \neq X_0^2$.

Proof. We follow the same idea as in the proof of lemma 2.3: let Z_t be the random walk on \mathbb{Z} with $Z_0 = 1$ and step distribution $\mathbb{P}(Z_{t+1} - Z_t = \pm 1) = 1/4$, $\mathbb{P}(Z_{t+1} - Z_t = 0) = 1/2$, so that

$$\tau_{2m} \stackrel{d}{=} B + (1 - B) \inf\{t > 0 : Z_t = 0 \text{ or } m\} \equiv B + (1 - B)(1 + \sigma_m), \quad (2.3)$$

where B is a Bernoulli(1/2) variable independent of everything. Thus τ_{2m} is closely related to the two-sided hitting time $\sigma_m = \text{first time to hit } 0 \text{ or } m \text{ for a random walk on } \mathbb{Z}$. The distribution of σ_m can be calculated by considering the clever martingale

$$M_t = \frac{\exp(rZ_t)}{\mathbb{E} \exp(rZ_t)} = \frac{e^{r(Z_t-1)}}{\cosh(r/2)^{2t}}.$$

Specifically, for any $r > 0$, $\{M_t, t \geq 0\}$ is a discrete time martingale with respect to itself, since the increments of Z are independent variables. The second equality above comes from calculating the expectation of the increments:

$$\mathbb{E} e^{r\Delta Z} = \frac{1}{4} e^r + \frac{1}{4} e^{-r} + \frac{1}{2} = \cosh(r/2)^2.$$

Note that $M_0 = 1$, and recall the well known fact that for a simple random walk \mathbb{S}_t started from 0,

$$\mathbb{P}(\mathbb{S}_t = -a | t = \inf\{s > 0 : \mathbb{S}_s \notin (-a, b)\}) = \frac{b}{a + b}$$

for $a, b > 0$. Z is not a simple random walk, but the step distribution for Z is symmetric, and Z takes steps ± 1 , so the same fact holds for Z . Thus by the optional stopping theorem,

$$1 = \mathbb{E}[M_{\sigma_m}] = \mathbb{E} \left[\frac{m-1}{m} \frac{1}{e^r \cosh(r/2)^{2\sigma_m}} + \frac{1}{m} \frac{e^{mr}}{e^r \cosh(r/2)^{2\sigma_m}} \right].$$

Doing some algebra, and substituting $w = \cosh(r/2)^{-2} \in (0, 1)$, yields

$$\mathbb{E}[w^{\sigma_m}] = \frac{m \left(\frac{1}{\sqrt{w}} - \sqrt{\frac{1}{w} - 1} \right)^2}{m - 1 + \left(\frac{1}{\sqrt{w}} - \sqrt{\frac{1}{w} - 1} \right)^{2m}}.$$

The result follows by plugging into the distributional equation (2.3). □

Note that finding the actual distribution of τ_{2m} would involve taking a discrete inverse Laplace transform of (2.2), which is a bit ugly! We can use (2.2) to explicitly compute some moments of τ_{2m} . Before we do so, we note that it is easy to directly compute the mean and variance of τ_{2m} when $m = 1$ or 2 . Clearly $\tau_2 \equiv 1$, while τ_4 is geometrically distributed with mean 2, since at each step there is always probability $1/2$ that the walkers jump to the same site on $\mathbb{Z}/4\mathbb{Z}$. When $m \geq 3$, there is no fixed probability of a collision at each step, so τ_{2m} is not sub-exponential in this case, and so its moments may not be well behaved. The next result confirms this intuition:

Corollary 2.5. $\mathbb{E}\tau_{2m} = m$ for all $m \geq 1$. Also, for $m \geq 3$, $\mathbb{E}\tau_{2m}^2 = \infty$.

Proof. For any random variable X , derivatives of the probability generating function are related to moments of X :

$$\begin{aligned} \frac{\partial}{\partial w} \Big|_{w=1} \mathbb{E}w^X &= \sum_{k \geq 1} \frac{\partial}{\partial w} \Big|_{w=1} \mathbb{P}(X = k)w^k \\ &= \sum_{k \geq 1} k\mathbb{P}(X = k) \\ &= \mathbb{E}X, \end{aligned}$$

and similarly,

$$\begin{aligned} \frac{\partial^2}{\partial w^2} \Big|_{w=1} \mathbb{E}w^X &= \sum_{k \geq 1} \frac{\partial^2}{\partial w^2} \Big|_{w=1} \mathbb{P}(X = k)w^k \\ &= \sum_{k \geq 1} k(k-1)\mathbb{P}(X = k) \\ &= \sum_{k \geq 1} k^2\mathbb{P}(X = k) - \sum_{k \geq 1} k\mathbb{P}(X = k) \\ &= \mathbb{E}X^2 - \mathbb{E}X. \end{aligned}$$

Thus by direct computation in (2.2),

$$\begin{aligned} \mathbb{E}\tau_{2m} &= \frac{\partial}{\partial w} \Big|_{w=1} \mathbb{E}[w^{\tau_{2m}}] = m, \\ \mathbb{E}\tau_{2m}^2 &= \frac{\partial^2}{\partial w^2} \Big|_{w=1} \mathbb{E}[w^{\tau_{2m}}] + \mathbb{E}\tau_{2m} = \begin{cases} 6, & m = 2 \\ \infty, & m \geq 3 \end{cases} \end{aligned}$$

□

Unlike the situation on \mathbb{Z} , the expected collision time is finite, so we can get a sense of how fast collisions are occurring. If $N_t(m)$ is the number of collisions of two particles on $\mathbb{Z}/2m\mathbb{Z}$ up to time t , then standard theorems about renewal processes (e.g. see [7]) imply

$$\frac{N_t(m)}{t} \rightarrow \frac{1}{m} \quad a.s.$$

and

$$\frac{\mathbb{E}N_t(m)}{t} \rightarrow \frac{1}{m}.$$

Another consequence of the formula in proposition 2.4 comes from letting $m \rightarrow \infty$, where we expect to recover the behavior of two random walks on \mathbb{Z} . The pointwise limit of $\mathbb{E}w^{\tau_{2m}}$ can be directly calculated. For $w \in (0, 1)$, $z = \frac{1}{\sqrt{w}} \in (1, \infty)$, and

$$\lim_{z \rightarrow 1^+} \frac{d}{dz} \left(z - \sqrt{z^2 - 1} \right) = -\infty.$$

It follows that

$$\frac{1}{\sqrt{w}} - \sqrt{\frac{1}{w} - 1} < 1$$

for all $w \in (0, 1)$, and thus

$$\lim_{m \rightarrow \infty} \mathbb{E}w^{\tau_{2m}} = \frac{w}{2} \left[1 + \left(\frac{1}{\sqrt{w}} - \sqrt{\frac{1}{w} - 1} \right)^2 \right] = \sum_{n=1}^{\infty} \frac{(2n-3)!!}{2^n n!} w^n, \quad (2.4)$$

where we have explicitly computed the series expansion of the limit. Recall the collision time τ from lemma 2.3. To conclude that (2.4) is the probability generating function of τ , we must show:

Proposition 2.6. *As $m \rightarrow \infty$, $\tau_{2m} \rightarrow_d \tau$.*

Proof. We must show that $\mathbb{P}(\tau_{2m} \leq k) \rightarrow \mathbb{P}(\tau \leq k)$ for each $k \in \mathbb{N}$, as $m \rightarrow \infty$. Note that probability that the random walk Z_t (defined in the proof of proposition 2.4) exits the interval $[0, m]$ at m is $\frac{1}{m}$, which converges to 0. Thus the random walk Z_t on the cycle $\mathbb{Z}/m\mathbb{Z}$ does not hit the point m with probability $\frac{m-1}{m}$, and in this case the probability of hitting 0 in time at most k is identical to the same probability for the corresponding walk on \mathbb{Z} . In other words, for any fixed $k > 0$,

$$\left| \mathbb{P}(\tau_{2m} \leq k) - \mathbb{P}(\tau \leq k) \right| \leq 1/m \rightarrow 0,$$

as desired. □

Thus we have:

Corollary 2.7. *The distribution of τ is explicitly given by*

$$\mathbb{P}(\tau = T) = \frac{(2T-3)!!}{2^T T!}.$$

(Note the convention $N!! = 1$ for $N \leq 0$.)

The case of 3 particles already is significantly more complicated. Let X_t^1, X_t^2, X_t^3 be simple random walks on \mathbb{Z} as above, with $X_0^1 = -2a, X_0^2 = 0$, and $X_0^3 = 2b$, with $a, b \geq 0$. We are interested in the first collision time

$$T(a, b) = \inf\{t > 0 : X_t^1 = X_t^2 \text{ or } X_t^2 = X_t^3\}.$$

We can actually compute the expected value of T , which is finite:

Proposition 2.8. $\mathbb{E}T(a, b) = 4ab$.

Proof. It turns out that this process is naturally related to a hitting time of a simple random walk on the hexagonal lattice. Let $U_t = \frac{X_t^2 - X_t^1}{2}$, $V_t = \frac{X_t^3 - X_t^2}{2}$, and let $\mathcal{H} = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \mp 1)\}$ denote the hexagonal lattice edges in \mathbb{Z}^2 , i.e. the usual lattice steps but with diagonals in one direction. It is easy to check that U and V have joint transition probabilities

$$\mathbb{P}(\Delta(U, V) = w) = 1/8 \text{ for } w \in \mathcal{H},$$

and

$$\mathbb{P}(\Delta(U, V) = 0) = 1/4,$$

where the increments

$$\Delta(U, V) \stackrel{d}{=} (U_{t+1}, V_{t+1}) - (U_t, V_t).$$

are independent. It follows that $T(a, b)$ is equal in distribution to the hitting time

$$\tau(a, b) = \inf\{t > 0 : W_t \in \partial Q\},$$

where W_t denotes the sleepy random walk (U_t, V_t) , $W_0 = (a, b)$, and $Q = \{(x, y) \in \mathbb{Z}^2 : x, y \geq 0\}$ (∂Q is the union of the positive x and y axes). Now, using theorem 3.5 in [13], the function $f : Q \rightarrow \mathbb{R}$ given by

$$f(v) = \mathbb{E}\tau(v)$$

is the minimal solution to the averaging equations

$$f(v) = 1 + \frac{1}{8} \sum_{w \in \mathcal{H}} f(v + w) + \frac{1}{4} f(v), \tag{2.5}$$

with the boundary conditions

$$f(v) = 0 \text{ for } v \in \partial Q. \tag{2.6}$$

It is easy to see that, writing $v = (a, b) \in \mathcal{H}$, the function $g(v) = 4ab1_{a>0}1_{b>0}$ is a solution to (2.5) and (2.6), since g clearly satisfies the boundary conditions, and

$$\begin{aligned} 1 + \frac{1}{8} \sum_{w \in \mathcal{H}} g(v + w) + \frac{1}{4} g(v) &= \frac{1}{4}(4ab) + \frac{1}{8} \left(4(a+1)b + 4(a-1)b + 4a(b+1) + \right. \\ &\quad \left. 4a(b-1) + 4(a-1)(b+1) + 4(a+1)(b-1) \right) \\ &= 1 + ab + 3ab + \frac{1}{8}(4 \cdot -2) \\ &= 1 + 4ab - 1 \\ &= g(v). \end{aligned}$$

To prove uniqueness, observe that for any two solutions g', g'' to the equations (2.5) and (2.6), the function $g'' - g'$ is harmonic on Q :

$$\frac{1}{6} \sum_{w \in \mathcal{H}} (g''(v+w) - g'(v+w)) = g''(v) - 8 - g'(v) + 8 = g''(v) - g'(v).$$

Moreover, by (2.6), $g'' - g'$ has boundary values 0, so since harmonic functions are determined by their boundary values [12], $g'' - g' \equiv 0$, or $g'' = g'$, as desired. \square

Alternative proof: Simply observe that

$$M_t = t + (X_t^2 - X_t^1)(X_t^3 - X_t^2) \quad (2.7)$$

is a martingale (a straightforward calculation). By the optional stopping theorem,

$$\mathbb{E}M_{\tau(a,b)} = \mathbb{E} \left[\tau + (X_\tau^2 - X_\tau^1)(X_\tau^3 - X_\tau^2) \right] = \mathbb{E}\tau = 4ab. \quad (2.8)$$

\square

The formula for the expected value was guessed after doing some simulations of the hexagonal walk process. It is tempting to think a simpler proof of this fact is possible: we have shown that the expected hitting time is simply the product of the initial inter-distances $X_0^2 - X_0^1 = 2a$ and $X_0^3 - X_0^2 = 2b$.

The idea in the proof of proposition 2.8 generalizes to multiple random walks: given n random walks $\{X_t^i\}$ for $1 \leq i \leq n$ and $\eta = (\eta^1, \eta^2, \dots, \eta^n) \in \mathbb{Z}^n$ satisfying $0 < \eta^1 < \eta^2 < \dots < \eta^n$, set

$$\tau(\eta) = \inf\{t > 0 : X_t^i = X_t^j, \text{ some } i \neq j \mid X_0^i = \eta^i \text{ for all } i\}$$

to be the first collision time given initial configuration η . Form the random vector

$$W_t = \left(\frac{X_t^2 - X_t^1}{2}, \frac{X_t^3 - X_t^2}{2}, \dots, \frac{X_t^n - X_t^{n-1}}{2} \right) \in \mathbb{Z}^{n-1},$$

which satisfies

$$\|W_{t+1} - W_t\|_1 \in \{0, 1, 2, \dots, n-1\}$$

for all $t \geq 0$. Suppose \mathcal{H}_n are the possible steps W_t can take, not including the possible sleepy step $W_{t+1} = W_t$, which happens with probability 2^{1-n} ; then just as in the proof of proposition 1.8, the expected first collision time $f(v) = \mathbb{E}\tau(v)$ must solve the system of equations

$$f(v) = 1 + \frac{1}{|\mathcal{H}_n|} \sum_{w \in \mathcal{H}_n} f(w+v) + 2^{1-n} f(v),$$

with boundary conditions

$$f(v) = 0 \text{ for } v \in \partial C_{n-1},$$

where C_d is the positive orthant in \mathbb{Z}^d , $C_d = \{x^1, x^2, \dots, x^d \geq 0\}$. If one actually wanted to solve this system in general, it would require determining exactly what the set \mathcal{H}_n is: this may take some work. For example, in the case $n = 4$ we have

$$\mathcal{H}_4 = \{(\pm 1, 0, 0), (0, 0, \pm 1), (\pm 1, \mp 1, 0), (0, \pm 1, \mp 1), (0, \pm 1, 0), (\pm 1, 0, \mp 1), (\pm 1, \mp 1, \pm 1)\}.$$

Then, one would have to come up with a candidate solution for the above equations, and it isn't obvious how to do so. As we will see later, this framework for analyzing collisions is analogous to the case for brownian motion: collisions of multiple particles in one dimension is equivalent to a hitting time of a single high dimensional random walk, with skew transition probabilities, or a non-trivial covariance matrix.

3 Biased walks

We now turn to the asymmetric case on \mathbb{Z} . Consider two random walks X_t^1, X_t^2 on \mathbb{Z} with biases p_1 and p_2 , respectively, i.e. $\mathbb{P}(\Delta X^i = 1) = p_i$ for $i = 1, 2$. Start both walks from $X_0^1 = X_0^2 = 0$. We aim to understand the stopping time

$$\tau_{p_1, p_2} = \inf\{t > 0 : X_t^1 = X_t^2\} \quad (3.1)$$

Consider the walk $Z_{p_1, p_2, t}$ given by

$$Z_{p_1, p_2, t} = X_t^1 - X_t^2, \quad (3.2)$$

and let \tilde{Z} be the non-sleepy version of Z , i.e. set

$$s = \frac{p_1 q_2}{p_1 q_2 + p_2 q_1}, \quad (3.3)$$

and

$$\mathbb{P}(\Delta \tilde{Z} = 1) = s, \quad \mathbb{P}(\Delta \tilde{Z} = -1) = 1 - s. \quad (3.4)$$

(Here $q_i = 1 - p_i$.) The key is to understand the stopping time

$$\sigma_s = \inf\{t > 0 : \tilde{Z}_t = 0\} \quad (3.5)$$

given that $\tilde{Z}_0 = 1$, and assuming $s \leq \frac{1}{2}$. Unlike in the unbiased case $s = \frac{1}{2}$, σ_s has finite expectation for $s < \frac{1}{2}$. To calculate the distribution of σ_s , note that on the event $\{\sigma_s = 2k + 1\}$ for $k \in \mathbb{N} \cup \{0\}$, \tilde{Z} travels a Dyck path starting and ending at $\tilde{Z}_0 = \tilde{Z}_{2k} = 1$, and then takes a -1 step from $\tilde{Z}_{2k} = 1$ to $\tilde{Z}_{2k+1} = 0$. Thus

$$\mathbb{P}(\sigma_s = 2k + 1) = s^k (1 - s)^{k+1} C_k, \quad (3.6)$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ is the k th Catalan number. (As an aside, this offers a bijective proof of the fact that

$$\sum_{k \geq 0} s^k (1 - s)^{k+1} \frac{1}{k+1} \binom{2k}{k} = 1$$

for $s \leq \frac{1}{2}$.) Some analysis yields the explicit formulas

$$\mathbb{E}\sigma_s = \frac{1}{1 - 2s}, \quad \mathbb{E}\sigma_s^2 = \frac{1 + 2s - 4s^2}{(1 - 2s)^3}.$$

4 The infinite collision property

Another take on the question of collisions is to try and get a result like proposition 2.1 on other underlying graphs. That is, if we have two simple random walkers on a graph G , do they collide infinitely often almost surely? Which graphs have this property? This is examined in depth in [1], which we give a short account of now. Given a graph G , let X^1, X^2 be independent random walks on G started from any vertex $v \in G$, and let \mathbb{P}_v denote the law of this process. Just as in our proposition 2.1, define

$$Z = \sum_{t=0}^{\infty} 1[X_t^1 = X_t^2].$$

We say G has the *infinite collision property* if

$$\mathbb{P}_v(Z = \infty) = 1 \quad \forall v \in G,$$

and G has the *finite collision property* if

$$\mathbb{P}_v(Z < \infty) = 1 \quad \forall v \in G.$$

The first important fact is a zero-one law:

(BPS proposition 2.1) *If G is a recurrent graph, then $\mathbb{P}_v(Z = \infty) \in \{0, 1\}$. Also, we have either $\mathbb{P}_v(Z = \infty) = 0$ for all v or $\mathbb{P}_v(Z = \infty) = 1$ for all v .*

Surprisingly, graphs can be recurrent, and yet also have the finite collision property. This is one of the motivations for the authors in [1]; see also [11]. Define the graph $Comb(\mathbb{Z})$ to have vertices $\mathbb{Z} \times \mathbb{Z}$ and edges

$$\{(x, n), (x, m)\} : |m - n| = 1\} \cup \{(x, 0), (y, 0)\} : |x - y| = 1\}.$$

A variant on the comb graph is $Comb(\mathbb{Z}, \alpha)$, for $\alpha > 0$, which is defined as the induced subgraph of $Comb(\mathbb{Z})$ with vertex set $\{(x, y) : 0 \leq y \leq x^\alpha\}$. Note that the comb graphs are recurrent. The comb graphs exhibit the desired phenomenon, and have a phase transition in terms of α . The main result in [1] is:

(BPS theorems 1.6 & 1.8) *If $\alpha \leq 1$, then $Comb(\mathbb{Z}, \alpha)$ has the infinite collision property; and if $\alpha > 1$, then $Comb(\mathbb{Z}, \alpha)$ has the finite collision property. Also, all the following graphs have the infinite collision property: a critical Galton-Watson tree with finite variance conditioned to survive forever; the incipient infinite cluster for critical percolation in dimension $d \geq 19$; the uniform spanning tree in \mathbb{Z}^2 .*

One of the keys to proving such theorems is a criterion in terms of the Greens function of the graph G that determines whether G has the infinite collision property:

(BPS theorem 3.1) *Let G be a recurrent graph with a distinguished vertex o , and let g_A denote the Green kernel of G with respect to a subset $A \subset G$. Let $\{B_r\}_r$ be a strictly increasing sequence of sets whose union is G . Suppose there exists $C < \infty$ such that for all r ,*

$$g_{B_r}(x, x) \leq C g_{B_r}(0, 0) \quad \forall x \in B_r.$$

Then G has the infinite collision property.

This allows the authors to use analytic information about the Greens functions of some graphs, like the Galton-Watson tree, to decide the collision property.

5 Collisions of brownian particles

Andrey Sarantsev's treatment of the situation with multiple Brownian particles[15] is typical of an approach to this problem. The set up is as follows: let W_1, W_2, \dots, W_N be standard Brownian motions in one dimension, and for a continuous \mathbb{R}^N -valued process

$$X = (X_1(t), \dots, X_N(t)),$$

let $p_t = p_{X(t)}$ denote the ranking permutation for the vector X , i.e. for every $t \geq 0$ we have

$$X_{p_t(1)} \leq X_{p_t(2)} \leq \dots \leq X_{p_t(N)}.$$

Then the process we seek to study satisfies the SDE

$$dX_i(t) = \sum_{k=1}^N 1[p_t(k) = i](g_k dt + \sigma_k dW_i(t)), i = 1, \dots, N, \quad (5.1)$$

where $g_k \in \mathbb{R}$ are the drift coefficients and $\sigma_k \geq 0$ are the 'diffusion' coefficients. This is a very appealing model, because it is like the particles 'bounce off' of each other when they collide: the i th largest particle always has the same drift and diffusion coefficients. To prove that this process exists and is unique – that is, (5.1) has a unique solution – it is necessary to assume no triple collisions occur. Andrey is able to prove:

(S15 Theorem 1.4) *If the sequence σ_n^2 is concave, i.e.*

$$\sigma_{k+1}^2 - \sigma_k^2 \leq \sigma_k^2 - \sigma_{k-1}^2, k = 2, \dots, N,$$

then with probability one, no triple collisions occur. Moreover, if the concavity condition fails for some k , then there exists a time $t > 0$ such that there is a triple collision between particles with ranks $k - 1, k$ and $k + 1$ at time t .

This is a beautiful necessary and sufficient condition for triple collisions. The proof involves studying a reflected Brownian motion in a high dimensional wedge. Specifically, just as we have in section 2, Andrey defines the 'gap process'

$$Z_k(t) = X_{k+1}(t) - X_k(t), k = 1, \dots, N - 1,$$

and shows that the gap process is a reflected Brownian motion with explicit drift and diffusion matrices given in terms of the g_k and σ_k (pages 8-9). This process is then analyzed by reducing to the two-dimensional wedge case, where Brownian processes are very well understood.

References

- [1] M. Barlow, Y. Peres, and P. Sousi. Collisions of random walks. *Annales de l'Institut Henri Poincaré*, Volume 48(4):922–946, 2012.
- [2] E. Ceba and D. Leping. No multiple collisions for mutually repelling brownian particles. *Seminaire de Probabilités XL*, 1899:241–246, 2005.
- [3] Z. Ciesielski and S. J. Taylor. First passage times and sojourn times for brownian motion in space and the exact hausdorff measure of the sample path. *Transactions of the American Mathematical Society*, 103(3):434–450, 1962.
- [4] C. Cooper, A. Frieze, and T. Radzik. multiple random walks in random regular graphs. *SIAM journal of discrete mathematics*, 23(4):1738–1761, 2009.
- [5] D. Coppersmith, P. Tetali, and P. Winkler. Collisions of random walks on a graph. *SIAM journal of discrete mathematics*, 6(3):363–374, 1993.
- [6] M. I. G. de Soria, P. Maynar, and E. Trizac. Brownian motion under annihilation dynamics. *Physical Review*, 78(6), 2008.
- [7] R. Durrett. Probability theory and examples. pages 208–210.
- [8] D. J. Grabiner. Brownian motion in a weyl chamber, non-colliding particles, and random matrices. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 35(2):177–204, 1999.
- [9] D. G. Hobson and W. Werner. Non-colliding brownian motions on the circle. *Bulletin of the London Mathematical Society*, 28(6):643–650, 1996.
- [10] T. Ichiba and I. Karatzas. On collisions of brownian particles. *The annals of applied probability*, 20(3):951–977, 2010.
- [11] M. Krishnapur and Y. Peres. Recurrent graphs where two independent random walks collide finitely often. *Electronic communications in probability*, Volume 9(8):72–81, 2004.
- [12] R. Lyons and Y. Peres. Probability on trees and networks. pages 19–23, 2017.
- [13] J. Norris. Markov chains. pages 9–12, 1997.
- [14] N. O'Connell and M. Yor. A representation for non-colliding random walks. *Electronic communications in probability*, 7:1–12, 2002.
- [15] A. Sarantsev. Triple and simultaneous collisions of competing brownian particles. *Electronic journal of probability*, 20(29):28–, 2015.