Stochastic abelian particle systems

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- Pick $v \in G$ uniformly at random with $\eta(v) \geq \tau$ (unstable site)
- Topple at v: τ particles at v each step to a uniform random neighbor of v, all independently.
- Write $\eta \to T_v \eta$ (toppling operator)

Absorbing state phase transition & self organized criticality

Robust w.r.t. initial conditions, toppling rules; variants

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Abelian property, site-wise representation, ‘chip firing’

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Forest fires, earthquakes, avalanches

Hyperuniformity, power laws

Driven-dissipative/SSM markov chain:

1. Any(!) initial configuration on $B_n = [-n, n]^d \subset \mathbb{Z}^d$
2. Perform SSM dynamics, only toppling sites in B_n
3. When all sites are stable ($< \tau$ particles), add a single particle to B_n
4. Return to step 2

Note: particles may step outside B_n. No mass conservation!

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Start with $\eta(v) \sim \text{Poisson}(\mu)$ independently over $v \in G$

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Local fixation

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What is the probability of fixation? How does it depend on μ?

$0-1$ law

For any particle density μ, $\mathbb{P}(\mathsf{SSM}(\mu) \text{ fixates}) \in \{0,1\}$.

Monotonicity

If SSM(μ) fixates almost surely for some μ, then it fixates almost surely for $\mu' < \mu$.

Phase transition

There is a critical density $\mu_c = \mu_c(G, \tau) \in [0, \tau]$ satisfying

$$\mathbb{P}(\text{SSM}(\mu) \text{ fixates}) = \begin{cases} 1, \mu < \mu_c \\ 0, \mu > \mu_c \end{cases}$$

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Concrete problems:

- Upper/lower bounds on μ_c?
- Time to fixate on a finite set?
- Order of the odometer function?
- Mixing time?

For SSM on $\mathbb{Z}$ with threshold $\tau = 2$:

$$\frac{1}{2} \leq \mu_c < 1$$

- Lower bound: Podder, Rolla '20
- Upper bound: Hoffman, Hu, R., Rizzolo '23

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Conjecture: for $\tau = 2$, $\mu_c(\mathbb{Z}^2) < 1$

Site-wise representation

Generate iid 'instructions' $\{\xi_{v,j}\}$ for $v \in G$ and $j \geq 1$

$\xi_{v,j}$ is uniform over $\{M_{v \to w} : w \text{ neighbor of } v\}$, where

$$M_{v \to w}\eta(x) = \begin{cases} \eta(x), & x \notin \{v, w\} \\ \eta(v) - 1, & x = v \\ \eta(w) + 1, & x = w \end{cases}$$

Apply $\xi_{v,j}$ at the jth time a particle topples at site v.

To run the dynamics on a finite subset, we choose a (legal) sequence of sites $x_1, \ldots, x_r$ to topple:

$$\eta \to T_{x_r} \cdots T_{x_1} \eta.$$

Issue: what if different toppling sequences give different results?

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v is *stable* for η if $\eta(v) < \tau$

If all v are stable, then we call η *stable*

If the result is stable, the order of topplings didn't matter!

Abelian property

Fix η, and any (legal) toppling sequences $T^x = (T_{x_1}, \ldots, T_{x_r})$ and $T^y = (T_{y_1}, \ldots, T_{y_s})$ such that $T^x\eta$ and $T^y\eta$ are stable. Then T^x is a permutation of T^y.

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Proof: 1) For unstable sites x and y, $T_x T_y \eta = T_y T_x \eta$

2) Find $y_k = x_1$, swap T_{y_k} to the front:

$$T_{y_1} \cdots T_{y_{k-1}} T_{x_1} T_{y_{k+1}} \cdots T_{y_s} \eta = T_{x_1} T_{y_1} \cdots T_{y_{k-1}} T_{y_{k+1}} \cdots T_{y_s} \eta$$

Repeat for all x_i. Since $T^x\eta$, $T^y\eta$ are stable, no unstable y's remain. □

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Remark: no randomness here. Holds for any fixed realization of the stacks.

We can choose clever toppling sequences, as long as we fully stabilize.

The toppling sequence can even be chosen as a function of the stack instructions.

ARW (activated random walk). Fix $\lambda > 0$ (sleep rate). Dynamics:

- Configuration $\eta : G \to \mathbb{N}^{\geq 0} \cup \{s\}$
- Particles are *active* or *sleepy*
- Active particles perform simple random walk at rate 1
- Sleepy particles (s) do not move

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- Particles are *active* or *sleepy*
- Active particles perform simple random walk at rate 1
- Sleepy particles (s) do not move
- Each active particle becomes sleepy at rate λ
- If $\eta(v) \geq 2$, all particles at v instantly become active

ARW is like SSM, but easier because of additional randomness

Same properties: SOC, phase transition, abelian

Interpolates between independent SRWs ($\lambda = 0$) and IDLA ($\lambda = \infty$)

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Selection of recent results:

- On $\mathbb{Z}$:
  - $\zeta_c(\lambda) \geq \frac{\lambda}{1+\lambda}$ (Rolla, Sidoravicius '12)
  - $\zeta_c(\lambda) \to 0$ as $\lambda \to 0$ (Basu, Ganguly, Hoffman '15)
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- On $\mathbb{Z}^d$, $d \geq 2$:
  - $\zeta_c(\lambda) < 1$ for λ small (Forien, Gaudilliere, '22; Hu, '23)
- Relaxation/mixing time (Bristiel, Salez '22; Levine, Liang '23)

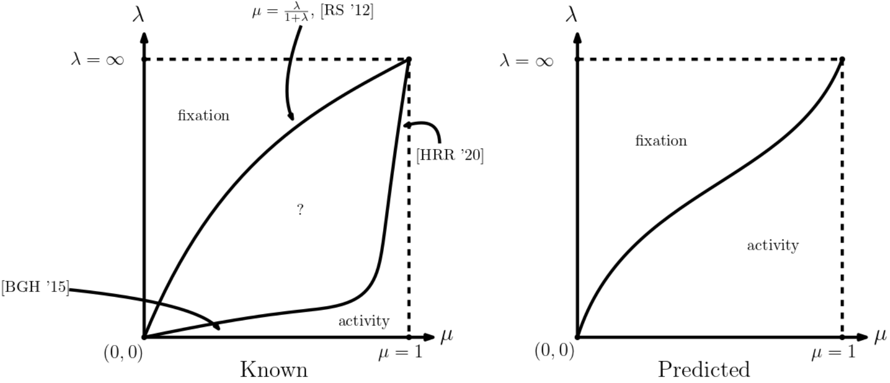

Figure: Phase diagram for ARW on $\mathbb{Z}$.

Rolla, Sidoravicius '12

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Rolla, Sidoravicius, Zindy '19

Fix $\lambda > 0$ and $G = \mathbb{Z}^d$. For any ergodic (active) initial configuration η with particle density ζ, ARW(ζ, λ) started from η fixates if $\zeta < \zeta_c$ and stays active if $\zeta > \zeta_c$.

'Critical density is universal'

Let x_k = position of kth particle to the right of 0, $k = 1, 2, \ldots$

Define the traps a_k recursively:

- $a_0 = 0$.
- For $k > 0$: send a ghost particle out from x_k, ignoring sleep instructions, until it hits a_{k-1}.
- a_k = leftmost site to the right of a_{k-1} where the second to last instruction seen by the ghost was a sleep instruction.

Particles follow the paths of their ghosts, except that they fall asleep in the trap.

Note: an instruction is a sleep instruction with probability $\frac{\lambda}{1+\lambda}$

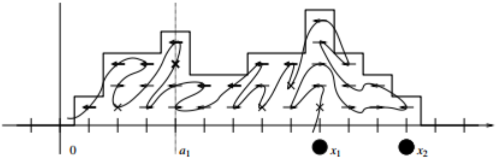

Figure: A diagram from [RS '12], showing the first trap a_1 for the particle x_1.

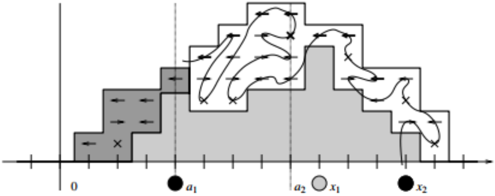

Figure: The trap a_2 for the particle x_2, obtained recursively by exploring the stack instructions.

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Thus if $\zeta < \frac{\lambda}{1+\lambda}$, $x_k > a_k$ for all k large a.s.

So $\mathbb{P}(\text{fixation}) > 0$. By the 0-1 law, $\mathbb{P}(\text{fixation}) = 1$. $\square$

Basu, Ganguly, Hoffman, R. '17

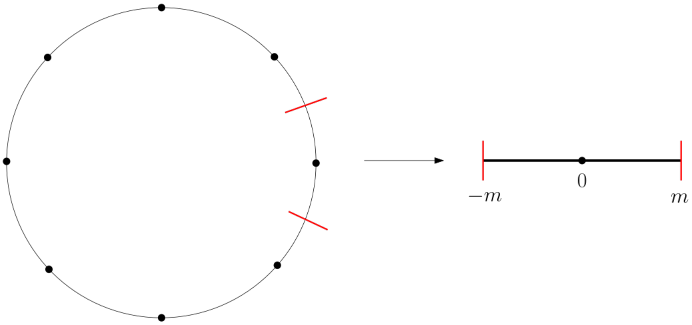
Consider ARW on $\mathbb{Z}/n\mathbb{Z}$. For any $\lambda \in (0, \infty]$ and $\zeta < \frac{\lambda}{1+\lambda}$,

$$\#\text{ stack instructions to fixate} < Cn\log(n)^2$$

with high probability as $n \to \infty$ for some $C > 0$.

The fixation speed depends on the initial condition: if all particles start at the same site, it takes at least Cn^3 instructions whp.

First step: gather $\log n$ particles at each of $\frac{n}{\log n}$ sites.



Focus on a single sub-interval.

How to adapt the traps for an interval?

Two-sided traps: ghosts start at 0, traps are set recursively at the boundary. Procedure fails if the traps reach 0.

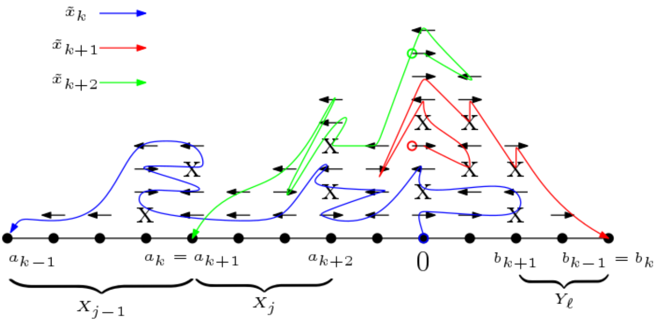


Figure: Setting traps 'in both directions' on an interval.

Internal erosion on an interval:

1. Start with the interval $X_0 = [-m, m] \cap \mathbb{Z}$.
2. Start a simple random walker from 0, stopped when she hits a boundary point $B \in \partial X_t$.
3. Remove the point B from X_0, to obtain $X_{t+1} = X_t \setminus \{B\}$.
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How large is the interval when the origin is eroded?

Idea: replace each segment $[j-1, j]$ and $[-j, -j+1]$ by an independent Exponential(j) length of rope, connect them all together, and initialize by lighting both ends on fire.

Properties of exponentials give a coupling between this process and the erosion process. Key computation: for $a, b > 0$,

$$\mathbb{P}^0(\text{hit } b \text{ before } -a) = \frac{a}{a+b} = \mathbb{P}(\text{Exp}(b) < \text{Exp}(a)).$$

(+ memoryless-ness)

Levine, Peres, '07

Let $R(m)$ be the number of sites remaining when the origin is eroded. Then

$$\frac{R(m)}{m^{3/4}} \to_d \left(\frac{8}{3}\right)^{1/4} \sqrt{|Z|},$$

as $m \to \infty$, where $Z \sim N(0,1)$.

Note: the number of remaining sites is $O(m^{3/4}) = o(m)$.

Issue: at each stage, one of the traps moves a random distance – distributed as Geo$\left(\frac{1+\lambda}{\lambda}\right)$ – not distance 1.

We are still able to couple with the rope process, but the exponentials have random means. Many concentration estimates necessary.

Conclusion: the left and right side traps still shrink to 0 at the same rate (up to lower order stuff). Two-sided trap setting succeeds for $\zeta < \frac{\lambda}{1+\lambda}$.