AI-GENERATED RESEARCH

Overlap geometry in higher dimensions

Entropy, forbidden patterns, and local conjugacies

Generated with ChatGPT from research questions posed by Jacob Richey.
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Forbid one small pattern in an infinite array. How much freedom remains? Does a pattern that overlaps itself more easily leave a larger avoidance space? And when do two forbidden patterns produce the same dynamical system, up to a reversible local change of coordinates?

This note selects results from an ongoing investigation. The new arguments have *working-proof status*; exact certificates support the stated finite enumerations, but no independent mathematical audit is claimed. Earlier results used as background are identified separately. Proofs here are sketches.

1 The geometry recorded by an agreement set

Let A be an alphabet of size $q \geq 2$, let $S \subset \mathbb{Z}^d$ be finite and nonempty, and let $w \in A^S$. The shift X_w consists of arrays in $A^{\mathbb{Z}^d}$ containing no translated copy of w. Define

$$G(w) = \{t \in (S - S) \setminus \{0\} : w(s) = w(s - t) \text{ for every } s \in S \cap (S + t)\}.$$

These are the displacements at which two copies can overlap consistently. The zero displacement and disjoint copies are excluded.

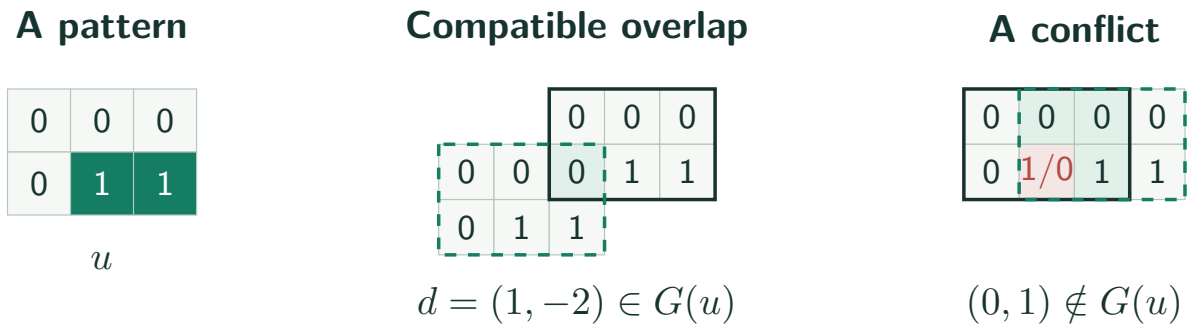


Figure 1. Two copies of $u = (000; 011)$. The green shared corner agrees. In the right panel, the entry $1/0$ records contradictory requests. Coordinates are (row, column), with rows increasing downward.

For a finite anchor set V, write I_z^w for the occurrence indicator at z, and $N_w = \sum_{z \in V} I_z^w$. For $J \subseteq V$, put $S + J = \bigcup_{z \in J} (S + z)$. Under uniform independent letters,

$$\mathbb{E} \prod_{z \in J} I_z^w = \begin{cases} q^{-|S+J|}, & \text{all overlapping pairs in } J \text{ are compatible,} \\ 0, & \text{otherwise.} \end{cases}$$

Pairwise consistency suffices because each event prescribes individual letters. Thus **equal G gives equal finite occurrence-field laws**: all mixed moments agree, and inclusion–exclusion recovers every atom of the law. Equal avoidance counts and entropy follow. This is the counting framework behind the earlier equal-agreement theorem of Chandgotia–Marcus–Richey–Wu [1]; it does not by itself construct a conjugacy.

2 The largest and smallest avoidance entropies

For $B_n = \{0, \dots, n-1\}^2$, let $L_w(B_n)$ count all A-labelings of B_n avoiding copies wholly contained in the box. We use free boundaries and natural logarithms:

$$h(X_w) = \lim_{n\to\infty} \frac{1}{|B_n|} \log L_w(B_n).$$

The entropy measures exponential growth per site, rather than the number of patterns in any one finite box.

Working result 1: both entropy extremes on every rectangle

Fix $S = \{0, \dots, a-1\} \times \{0, \dots, b-1\}$, $a, b \geq 2$, and $q \geq 2$. Constant patterns are the unique entropy maximizers, up to alphabet relabeling. If $G = \varnothing$ is realizable, it is exactly the minimizing agreement class. For $q \geq 3$ it is always realizable. For $q = 2$ it is realizable exactly when $\max(a, b) \geq 4$.
On the exceptional binary shapes 2×2, 2×3, 3×3, and their transposes, the minimizers have exactly one corner-agreement pair: $G = \{\pm(a-1, b-1)\}$ or $\{\pm(a-1, 1-b)\}$.

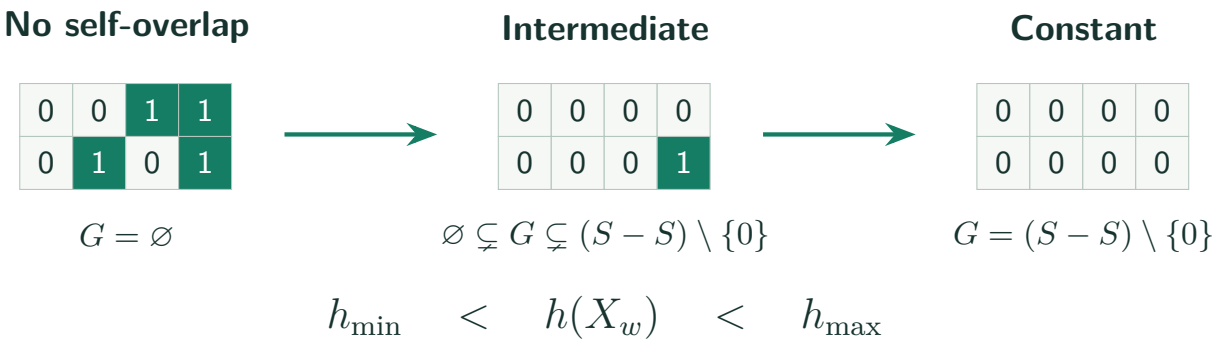


Figure 2. Exact binary 2×4 examples at the lower extreme, at an intermediate class, and at the upper extreme. The arrows record strict inequalities, not numerical estimates of the gaps.

Proof sketch. For the maximum, *merge letters*. If $v = f \circ u$, expose all sites except one. Among occurrence requests already satisfied elsewhere, let r, s count those asking for two letters that will merge, and let C count occurrences not involving this site. For a convex test function ϕ, the conditional comparison is

$$\phi(C + r) + \phi(C + s) \;\leq\; \phi(C) + \phi(C + r + s).$$

Iterating gives $\mathbb{E}\phi(N_u) \leq \mathbb{E}\phi(N_v)$ for every convex ϕ. Taking $\phi(n) = (1-n)_+$ gives non-strict avoidance order. Every pattern can be merged to a constant.

Strictness needs more. Choose a newly compatible pair of copies and process a surrounding collar before resolving their last conflict. A letter unused by the merged requests blocks unwanted copies through the collar. For every extendible exterior, the final merge creates at least one extra avoiding filling. A positive density of separated collars gives an exponential gain, hence a strictly positive entropy gap. Convexity of the rectangular support ensures that a suitable conflict site lies in every copy contained in the witness pair's union.

For the minimum, an empty-G pattern has mutually exclusive overlapping occurrences. The Shearer avoidance recursion gives the non-strict lower bound. A maximal compatible pair creates a quantitative defect in that recursion; a conditional local-lemma estimate makes the defect uniform in volume. Exact rational estimates handle the small remaining rectangles. The three exceptional binary shapes require separate finite-class arguments.

3 A partial order, with a substantial settled range

The central entropy conjecture is

$$G(u) \subsetneq G(v) \quad \Longrightarrow \quad h(X_u) < h(X_v) \qquad \text{for a fixed rectangle and alphabet.}$$

The general statement is open. The current working proofs settle the following range with exact structural certificates.

Working result 2: strict order through area 25

If S is a two-dimensional rectangle with both sides at least two, $|S| \le 25$, and $q \ge 3$, then every proper agreement inclusion gives $h(X_u) < h(X_v)$. This includes patterns using all q symbols.
At $q = 2$, all these inclusions have $h(X_u) \le h(X_v)$, and strict pressure order at every finite nonzero coupling. Strict intermediate entropy gaps remain unresolved in general, even in this size range.

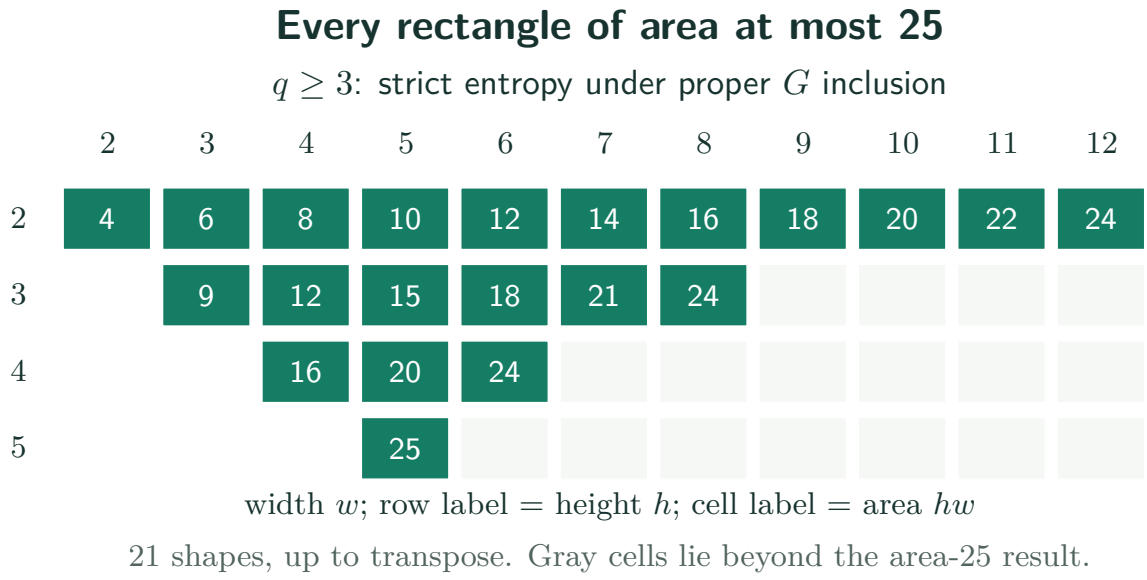


Figure 3. The finite range of supports in the theorem. The conclusion concerns infinite-volume entropy on each support; it is not an extrapolation from finite simulation boxes.

Proof sketch. Agreement sets form a finite inclusion poset. Joining sites forced equal by an agreement gives a component partition; its colorings describe which alphabets realize the class. It suffices to compare covers, that is, inclusions with no realizable class strictly between them.

If an upper class has a representative using fewer than q letters, split one forced component off using a fresh letter. The cover condition forces the new pattern into the lower class. Merging back reduces this comparison to the preceding theorem. The remaining covers through area 25 have literal certificates: 30 788 critical ternary covers and 74 critical four-symbol covers. Shared-letter, shared-layer, and row-sweep arguments give the required entropy gains. Their collars produce uniform local gains which can be repeated at positive density; finite-volume strictness alone would not suffice. Larger alphabets reduce to mergers.

An unbounded-size theorem. For a finite lattice-convex support $S = \mathrm{conv}(S) \cap \mathbb{Z}^d$, proper inclusion gives strict entropy if

$$\binom{q}{2} \ge D, \qquad D = \frac{|S - S| - 1}{2}.$$

For an $a \times b$ rectangle, $D = 2ab - a - b$. In a minimal-letter realization, every pair of letters must enable a new agreement when merged, and different pairs enable disjoint sets of new displacements. This bounds the number of necessary colors. On a strict inclusion chain, every upper class then has a spare color, so the cover-merger argument applies.

4 Why incomparable patterns need not have a fixed order

Let $Z_{w,n}(\beta) = \sum_{x \in A^{B_n}} \exp(\beta N_w(x))$, with anchors restricted to copies contained in B_n, and define the unnormalized pressure

$$P_w(\beta) = \lim_{n\to\infty} |B_n|^{-1} \log Z_{w,n}(\beta).$$

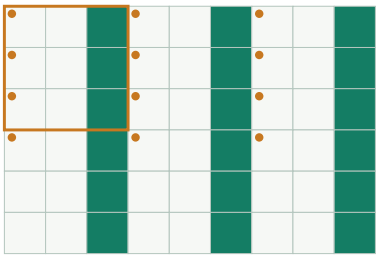
Here $P_w(0) = \log q$; positive β rewards occurrences, while negative β penalizes them. Entropy is the avoidance endpoint. A statistic useful near zero is

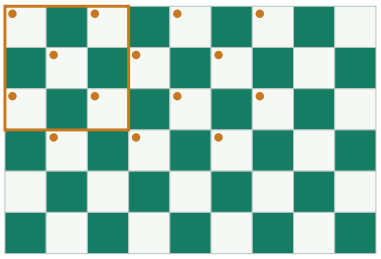
$$K_w(q) = \sum_{t \in G(w)} q^{|S \cap (S+t)|}.$$

Working result 3: an exact positive-pressure crossing

For $u = (001; 001; 001)$ and $v = (010; 101; 010)$ on a 3×3 support, $P_u(\beta) > P_v(\beta)$ for all sufficiently small $\beta > 0$, but $P_v(\beta) > P_u(\beta)$ whenever $\beta > 6 \log q$, for every $q \geq 2$. Their agreement sets are incomparable.

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad v = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$





$K_u(2) = 144, \qquad \rho_{\max}(u) = 1/3$ $\qquad$ $K_v(2) = 104, \qquad \rho_{\max}(v) = 1/2$

Periodic constructions attain these densities. Orange dots mark visible occurrence anchors.

Figure 4. Exact periodic arrays, not sampled configurations. White and green denote 0 and 1. The outlined 3×3 blocks are the forbidden patterns. Near-zero pressure and maximal packing favor opposite members of this pair.

Proof sketch. The overlap sets give

$$K_u = 2q^6 + 2q^3, \qquad K_v = 4q^4 + 4q^3 + 4q, \qquad K_u > K_v \quad (q \geq 2).$$

The first pressure derivatives agree. The covariance formula gives $P_u''(0) - P_v''(0) = q^{-18}(K_u - K_v) > 0$, so u has larger pressure near zero. This uses the analytic expansion around the independent product background.

The maximal occurrence densities go the other way:

$$\rho_{\max}(u) = \frac{1}{3}, \qquad \rho_{\max}(v) = \frac{1}{2}.$$

In any row, starts of 001 are separated by at least three columns; horizontally adjacent checkerboard-pattern occurrences are incompatible. The periodic constructions in Figure 4 attain the resulting bounds. Therefore

$$P_u(\beta) \leq \log q + \frac{\beta}{3}, \qquad P_v(\beta) \geq \frac{\beta}{2}.$$

These bounds reverse the near-zero order for $\beta > 6 \log q$. Continuity forces a positive crossing, without computing pressure numerically.

This example limits a proposed *total* ordering of patterns. It does not contradict ordering under agreement inclusion. A separate earlier 2×3 example, $u = (001; 011)$ and $v = (000; 101)$, has equal K and $P_v - P_u = q^{-16}\beta^3 + O(\beta^4)$; there too the agreement sets are incomparable.

5 A reversible local swap in every dimension

Equal entropy is much weaker than conjugacy. A conjugacy is a bijection commuting with every lattice translation, given by a finite local rule whose inverse is also local. One replacement mechanism has no dimensional restriction.

For $a, b \in A^S$, define the oriented cross-agreement set

$$C(a,b) = \{t \in (S-S) \setminus \{0\} : a(s) = b(s-t) \text{ for all } s \in S \cap (S+t)\}.$$

Thus $C(a,a) = G(a)$ and $C(b,a) = -C(a,b)$.

Working result 4: simultaneous-swap conjugacy

Let $S \subset \mathbb{Z}^d$ be any finite nonempty support and $u \neq v$ in A^S. If

$$C(u,u) = C(u,v) = C(v,u) = C(v,v),$$

then simultaneously exchanging every occurrence of u and v defines an involutive automorphism F of the full shift. Its coding neighborhood is contained in $S - S$, and F restricts to a conjugacy $X_u \to X_v$.

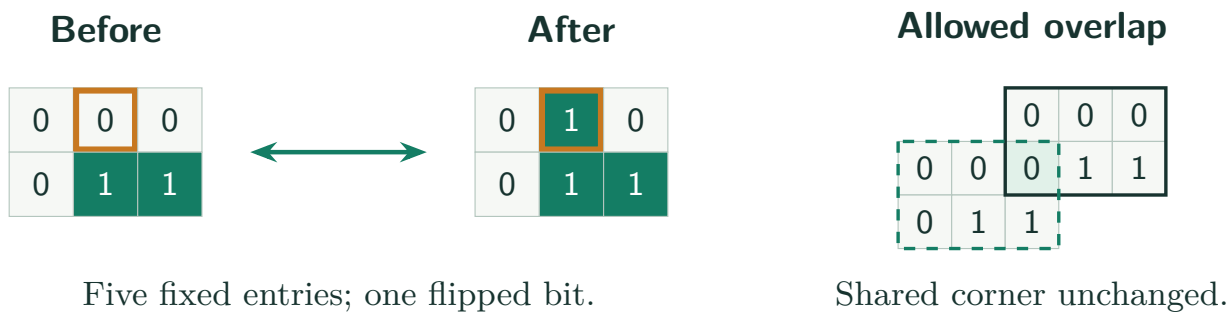


Figure 5. A binary 2×3 swap: $(000;011) \leftrightarrow (010;011)$. All four overlap sets are $\{\pm(1,-2)\}$. The rule acts simultaneously wherever the five fixed entries occur.

Proof sketch. For a common allowed displacement t and an overlapping site s, the four compatibility conditions imply

$$u(s) = v(s) = u(s-t) = v(s-t).$$

Thus every overlap position is unchanged by the swap. Replacements are consistent, and each old occurrence becomes the opposite type at its old anchor.

Suppose a new occurrence appeared. Some site in it must have changed, and that change belongs to an old occurrence which remains present after the swap. The new and old occurrences coexist, so their displacement is in the common cross-agreement set. Their shared site must therefore be unchanged, a contradiction. Occurrence anchors are preserved exactly. Swapping twice restores every letter. Computing the output at a site only requires checking occurrences covering it, all within its translate of $S - S$.

This proves the all-dimensional statement directly. Cycles and infinite clusters of occurrences require no cluster search. It extends the mechanism of the one-dimensional swap criterion in [1, Proposition 4.11]; no claim of a new underlying principle is intended.

Exact audit of the displayed rule. All $2^{15} = 32\,768$ input contexts needed to compute $F^2(x)_{0,0}$ and the output pattern at an anchor were enumerated. Each verifies both the involution and exchange of occurrence types. This checks the concrete rule; the general theorem rests on the proof.

6 Within a G-class: the next question

Open target, not a conclusion of this note

For a fixed support S and alphabet A, does

$$G(u) = G(v) \quad \Longrightarrow \quad X_u \cong X_v$$

hold? The main interest here is rectangular supports in higher dimensions. Equal G already gives equal occurrence laws and entropy. A general local conjugacy, or a counterexample, remains to be found. No converse is asserted.

The theorem on the preceding page requires *cross*-agreements as well as self-agreements. Equal G alone does not validate direct replacement. For $u = (00; 01)$ and $v = (10; 00)$, both have $G = \{\pm(1, -1)\}$, but replacing the sole v in the configuration below creates another v.

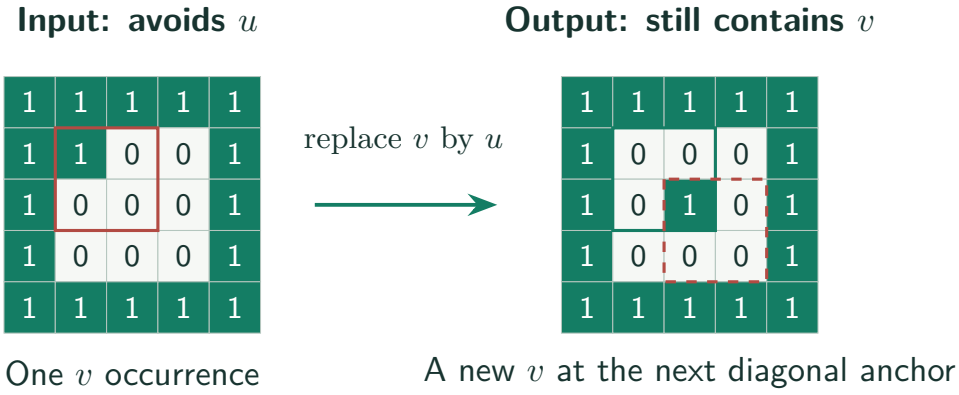


Figure 6. An exact obstruction to naive replacement. Red outlines mark v; the green outline marks the inserted u. This is a failure of the map, not a proof that the two avoidance shifts are nonconjugate.

A small positive classification. Exhaustively enumerating all 81 ternary 2×2 patterns gives seven agreement classes. Each class is connected by valid swaps and global alphabet permutations. Consequently equal G *does imply conjugacy for ternary* 2×2 *patterns*. For binary 2×2, two of six classes are disconnected under these particular moves; larger block codes remain possible.

Beyond the longest border. In one dimension, proper borders recurse through one longest proper border. In higher dimensions, a horizontal overlap strip and a vertical overlap strip can be incomparable. A proposed replacement is the family of actual overlap restrictions $(t, w|_{S\cap(S+t)})$, organized by containment while retaining translations and letters. This is an organizing idea, not yet a conjugacy construction. Any eventual decoder must use a uniformly bounded neighborhood.

Proofs, evidence, and reproducibility. The companion working record is `entropy-ordering.md`. The exact audits used here include `minimum_certificates.py`, `alphabet_sweep_checks.py`, `layer_strictness_checks.py`, `ternary_layer_checks.py`, `row_entropy_checks.py`, and `swap_conjugacy_checks.py`, with their saved JSON certificates. `build_illustrated_note.py` reproduces the PDF and all six vector figures; it checks the pictured agreement sets, overlap statistics, and shape count. Diagrams of periodic arrays are exact constructions. No numerical pressure curve is presented.

Earlier background

[1] N. Chandgotia, B. Marcus, J. Richey, and C. Wu. *Shifts of Finite Type Obtained by Forbidding a Single Pattern*. arXiv:2409.09024 (2024). Theorem 5.4 gives the earlier equal-agreement counting result; Proposition 4.11 states the one-dimensional swap criterion.